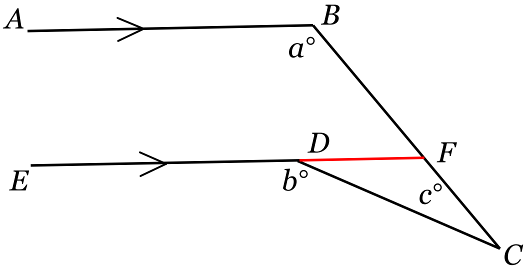
Adding auxiliary lines: Solutions

http://topdrawer.aamt.edu.au/Geometric-reasoning/Misunderstandings/Revealing-the-invisible/Adding-auxilliary-lines

1. **Aim:** To prove $b = a + c$

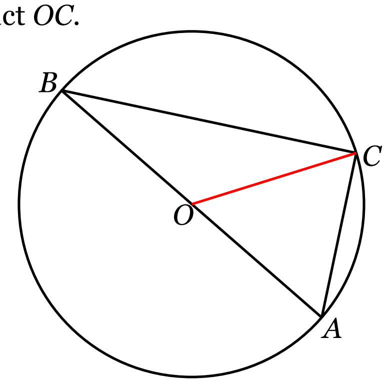
   Construction: Produce *ED* to meet *BC* at *F*.



   Proof:
   $\angle EFC = a°$ (corresponding angles, *ABPEF*)
   but $\angle EDC = \angle DFC + \angle FCD$ (exterior angle of *DCDF*)
   $\therefore\ b = a + c$

2. **Aim:** To prove $\angle ACB = 90°$

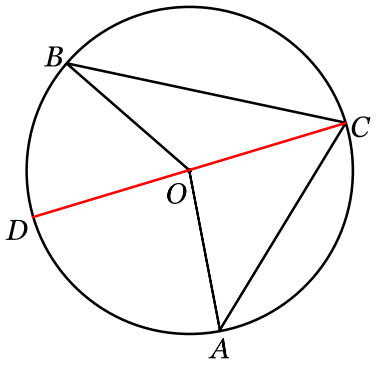
   Construction: Construct *OC*.



   Proof:
   $OA = OB = OC$ (radii)
   $\angle BCO = \angle OBC$ (opposite equal sides in ΔOBC)
   $\angle ACO = \angle CAO$ (opposite equal sides in ΔOAC)
   Now in ΔABC
   $2 \times \angle ACO + 2 \times \angle BCO = 180°$ (angle sum of ΔABC)
   $\therefore\quad \angle ACB = \angle ACO + \angle BCO$ (adjacent angles)
   $= 90°$
   $\therefore\quad \angle ACB = 90°$

3. **Aim:** To prove $\angle AOB = 2 \times \angle ACB$.

Construction: Construct the diameter CD.



Proof:

$OA = OB = OC$ (radii)

$\angle BCO = \angle OBC$ (opposite equal sides in ΔOBC)

$\angle ACO = \angle CAO$ (opposite equal sides in ΔOAC)

Now in ΔOBC:

$\angle BOD = \angle BCO + \angle OBC$ (exterior angle of ΔOBC)

$= 2 \times \angle BCO$ (1)

Similarly in ΔOAC:

$\angle AOD = 2 \times \angle ACO$ (2)

But $\angle AOB = \angle AOD + \angle BOD$ (adjacent angles)

$= 2 \times \angle ACO + 2 \times \angle BCO$ (from 1 and 2)

$= 2 \times (\angle ACO + \angle BCO)$

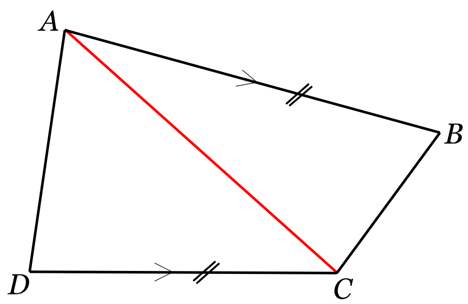
$= 2 \times \angle ACB$ (adjacent angles)

$\therefore \angle AOB = 2 \times \angle ACB$

4. **Data:** $AB = DC$ and $AB \parallel DC$.

Aim: Prove that $ABCD$ is a parallelogram.

Construction: Construct the diagonal AC.



Proof:

In ΔABC and ΔDAC

1. AC is common
2. $\angle BAC = \angle ACD$ (alternate angles, $AB \parallel DC$)
3. $AB = DC$ (data)

$\therefore\ \Delta ABC \equiv \Delta CDA$ (SAS)

$\therefore\ \angle BCA = \angle DAC$ (matching angles of congruent triangles)

But these are alternate angles

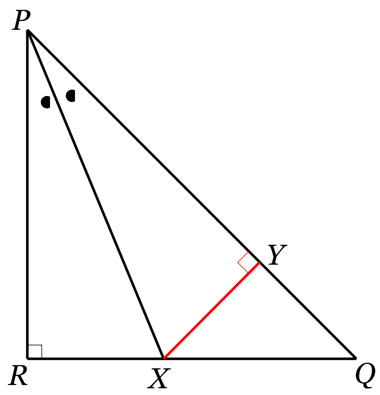
$\therefore\ AD \parallel BC$ (alternate angles are equal)

$\therefore$ *ABCD* is a parallelogram (two pairs of opposite sides parallel)

5. **Data:** $\angle PRQ = 90°$; $PR = RQ$; PX bisects $\angle RPQ$

Aim: Prove that $PQ = PR + RX$.

Construction: Construct the perpendicular from X to PQ meeting PQ at Y.



Proof:

In ΔPRX and ΔPYX

1. $\angle RPX = \angle XPY$ (PX bisects $\angle RPQ$)
2. $\angle PRX = \angle PYX = 90°$ (data)
3. PX is common

$\therefore\ \Delta PRX \equiv \Delta PYX$ (AAS)

$\therefore\ PR = PY$ (matching sides of congruent triangles) (i)

$\therefore\ RX = YX$ (matching sides of congruent triangles) (ii)

Now in ΔPQR:

$PR = RQ$ (equal sides of isosceles ΔPQR)

and $\angle PRQ = 90°$ (given)

$\therefore\ \angle PQR = 45°$ (property of right-angles isosceles triangle)

In ΔXYQ:

$\angle YXQ + 45° + 90° = 180°$ (angle sum of ΔXYQ)

$\therefore\ \angle YXQ = 45°$

$\therefore\ XY = YQ$ (opposite equal angles in ΔXYQ) (iii)

Now $PQ = PY + YQ$ (PYQ collinear)

$= PR + YQ$ (from i)

$= PR + XY$ (from ii)

$= PR + RX$ (from iii)

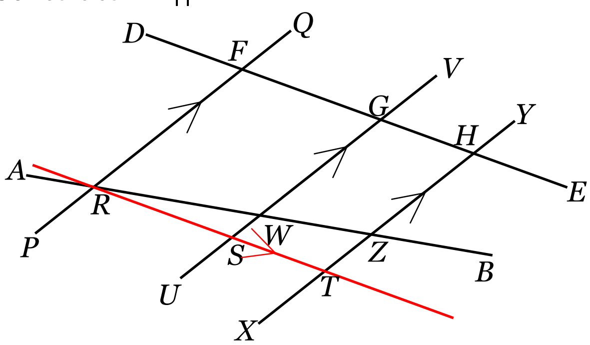
$\therefore\ PQ = PR + RX$

Challenge solutions

1. **Data:** $PQ \mid\mid UV \mid\mid XY$

Aim: Prove that $FG : GH = RW : WZ$

Construction: Construct $RT \mid\mid DE$



Proof:

In ΔRSW and ΔRTZ

1. $\angle RSW = \angle RTZ$ (corresponding angles, $UV \parallel XY$)
2. $\angle RWS = \angle RZT$ (corresponding angles, $UV \parallel XY$)

$\therefore \Delta RSW \,|||\, \Delta RTZ$ (AAA)

$\therefore RS : RT = RW : RZ$ (matching sides of similar triangles)

$\therefore RT : RS = RZ : RW$

$RZ = RW + WZ$ (R, W, Z collinear points)

Similarly, $RT = RS + ST$

Now $\frac{RT}{RS} = \frac{RZ}{RW}$

becomes $\frac{RS + ST}{RS} = \frac{RW + WZ}{RW}$

$\therefore \quad 1 + \frac{ST}{RS} = 1 + \frac{WZ}{RW}$

$\therefore \quad \frac{ST}{RS} = \frac{WZ}{RW}$

i.e. $ST : RS = WZ : RW$

$\therefore \quad RS : ST = RW : WZ$

RSGF is a parallelogram (2 pairs of opposite sides parallel)

$\therefore RS = FG$ (opposite sides of parallelogram)

Similarly, *STHG* is a parallelogram and $GH = ST$.

$\therefore FG : GH = RS : ST$

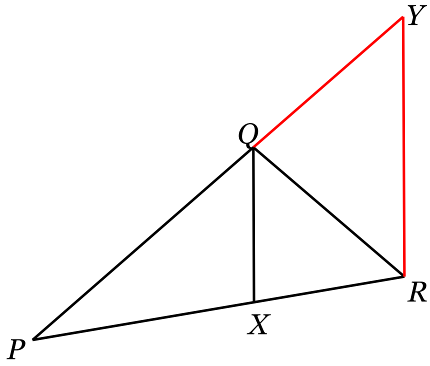
$= RW : WZ$

Note: This result is the theorem: Parallel lines preserve ratios of intercepts on transversals. It can be abbreviated as 'parallel lines preserve ratios'.

2. **Data:** QX bisects $\angle PQR$

Aim: Prove that $\frac{PQ}{QR} = \frac{PX}{XR}$.

Construction: Produce PQ to Y so that $QX \,||\, YR$.



Proof:
From Question 1, as $QX \,||\, YR$ then $PQ : QY = PX : XR$

Now $\angle PQX = \angle QYR$ (corresponding angles, $QX \,||\, YR$)

and $\angle XQR = \angle QRY$ (alternate angles, $QX \,||\, YR$)

But $\angle PQX = \angle XQR$ (QX bisects $\angle PQR$)
$\therefore \angle QYR = \angle QRY$
$\therefore \Delta QRY$ is isosceles
$\therefore QR = QY$ (opposite equal angles in ΔQRY)

$\therefore \frac{PQ}{QY} = \frac{PX}{XR}$ (parallel lines preserve ratios) i.e. the result from Qu. 1.

But $QY = QR$ (proven above)

$\therefore \frac{PQ}{QR} = \frac{PX}{XR}$

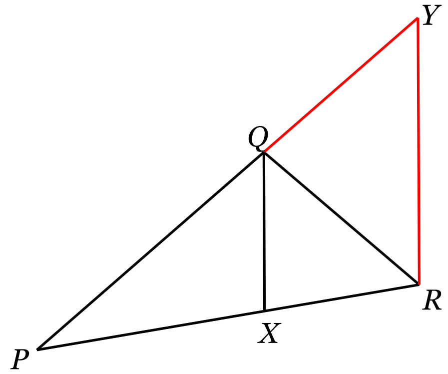
Note: This result is a corollary of the theorem proven in Question 1. An alternative proof, which does not use the theorem, appears on the following page.

Alternatively:

Data: QX bisects $\angle PQR$

Aim: Prove that $\frac{PQ}{QR} = \frac{PX}{XR}$.

Construction: Produce PQ to Y so that $QX \,||\, YR$.



Proof:

Let $\angle PQX = x°$
Then $\angle RQX = x°$ (QX bisects $\angle PQR$)
$\angle QYR = \angle QRY$ (opposite equal sides in isosceles ΔQYR)
$\angle QYR + \angle QRY = \angle PQR$ (exterior angle of ΔQYR)
$\therefore\ 2 \times \angle QYR = 2x°$
$\angle QYR = x°$
$\therefore\ \angle QYR = \angle PQX$

In ΔPQX and ΔPYR

1. $\angle P$ is common
2. $\angle PQX = \angle QYR$ (proven above)

$\therefore\ \Delta PQX \,|||\, \Delta PYR$ (AAA)

$\therefore\ \frac{PQ}{PY} = \frac{QX}{YR} = \frac{PX}{PR}$ (matching sides of similar triangles)

Taking $\frac{PQ}{PY} = \frac{PX}{PR}$:

$PQ \times PR = PX \times PY$
$PQ(PX + XR) = PX(PQ + QY)$ (PXR and PQY are collinear)
$PQ \times PX + PQ \times XR = PX \times PQ + PX \times QY$
$\therefore\ PQ \times XR = PX \times QY$

But $QY = QR$ by construction

$\therefore\ PQ \times XR = PX \times QR$

$\therefore\ \frac{PQ}{QR} = \frac{PX}{XR}$ (dividing throughout by $QR \times XR$)